Math 31 – Homework 6 Solutions

1. We will see in class that the kernel of any homomorphism is a normal subgroup. Conversely, you will show that any normal subgroup is the kernel of some homomorphism. That is, let G be a group with N a normal subgroup of G, and define a function $\pi : G \to G/N$ by

$$\pi(g) = Ng$$

for all $g \in G$. Prove that π is a homomorphism, and that $\ker \pi = N$.

Proof. We first check that π is a homomorphism. Let $g, h \in G$, and observe that

$$\pi(gh) = Ngh = (Ng)(Nh) = \pi(g)\pi(h).$$

Therefore, π is a homomorphism. Now we just need to identify its kernel. Suppose that $x \in \ker \pi$, so that

$$\pi(x) = Ne.$$

Since $\pi(x) = Nx$ by definition, this tells us that $Nx = Ne$. These two cosets are equal if and only if $xe^{-1} \in N$, or $x \in N$. Therefore, $x \in \ker \pi$ if and only if $x \in N$, so $\ker \pi = N$. □

2. Recall that $\mathbb{R}^\times$ is the group of nonzero real numbers (under multiplication), and let $N = \{-1, 1\}$. Show that N is a normal subgroup of $\mathbb{R}^\times$, and that $\mathbb{R}^\times/N$ is isomorphic to the group of positive real numbers under multiplication. [**Hint:** Use the Fundamental Homomorphism Theorem.]

Proof. Let $\mathbb{R}^\times_+$ denote the group of positive real numbers under multiplication. Define a map $\varphi : \mathbb{R}^\times \to \mathbb{R}^\times_+$ by

$$\varphi(a) = |a|.$$

Then φ is a homomorphism, since given $a, b \in \mathbb{R}^\times$,

$$\varphi(ab) = |ab| = |a||b| = \varphi(a)\varphi(b).$$

Furthermore, it is onto, since if a is a positive real number, $\varphi(a) = a$. Finally,

$$\ker \varphi = \{a \in \mathbb{R}^\times : |a| = 1\} = \{-1, 1\} = N.$$

Therefore, N is a normal subgroup of $\mathbb{R}^\times$. (This is also true simply because $\mathbb{R}^\times$ is abelian.) Furthermore, the Fundamental Homomorphism Theorem implies that

$$\mathbb{R}^\times/\{-1, 1\} \cong \mathbb{R}^\times_+.$$

□

3. [Saracino, #13.1] Let $\varphi : \mathbb{Z}_8 \to \mathbb{Z}_4$ be given by

$$\varphi(x) = [x]_4,$$

i.e., the remainder of x mod 4. Find $\ker \varphi$. To which familiar group is $\mathbb{Z}_8/\ker \varphi$ isomorphic?

Solution. Note that if $x \in \mathbb{Z}_8$, then $\varphi(x) = 0$ means that $[x]_4 = 0$, so 4 must divide x. Therefore, the kernel of φ is precisely the set

$$\ker \varphi = \{0, 4\}.$$

Since this subgroup has two elements, $\mathbb{Z}_8/\ker \varphi$ must have order 4. Therefore, it is isomorphic to either $\mathbb{Z}_4$ or the Klein 4-group. We claim that it is $\mathbb{Z}_4$—note that the coset $\ker \varphi + 1$ generates the quotient group:

$$\ker \varphi + 1 = \{1, 5\}(\ker \varphi + 1) + (\ker \varphi + 1) \qquad = \ker \varphi + 2 = \{2, 6\}$$
$$(\ker \varphi + 2) + (\ker \varphi + 1) = \ker \varphi + 3 = \{3, 7\}$$
$$(\ker \varphi + 3) + (\ker \varphi + 1) = \ker \varphi + 4 = \ker \varphi + 0 = \{0, 4\},$$

and these are all the cosets. Thus $\mathbb{Z}_8/\ker \varphi$ is a cyclic group of order 4, so it is isomorphic to $\mathbb{Z}_4$.

4. If G is a group and $M \trianglelefteq G$, $N \trianglelefteq G$, prove that $M \cap N \trianglelefteq G$. [You proved on an earlier assignment that $M \cap N$ is a subgroup of G, so you only need to prove that it is normal.]

Proof. As mentioned above, recall that $M \cap N$ is a subgroup of G. Let $a \in G$; we need to verify that $a(M \cap N)a^{-1} \subseteq M \cap N$. Let $g \in M \cap N$. Then $g \in M$, so $aga^{-1} \in M$ as M is normal. Similarly, $g \in N$, and $aga^{-1} \in N$ since N is normal. Therefore, $aga^{-1} \in M \cap N$. Since g was arbitrary, we have shown that $a(M \cap N)a^{-1} \subseteq M \cap N$, so $M \cap N \trianglelefteq G$.

5. Classify all abelian groups of order 600 up to isomorphism.

Solution. The first thing that we need to do is to factor 600 into a product of primes:

$$600 = 2^3 \cdot 3 \cdot 5^2.$$

The possible abelian groups of order 8 are:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$
$$\mathbb{Z}_2 \times \mathbb{Z}_4$$
$$\mathbb{Z}_8.$$

There is only one abelian group of order three, namely $\mathbb{Z}_3$, and for 25 there are two possibilities:

$$\mathbb{Z}_5 \times \mathbb{Z}_5$$
$$\mathbb{Z}_{25}.$$

Therefore, the abelian groups of order 600, up to isomorphism, are

$$\begin{aligned}
&\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\
&\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{25} \\
&\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\
&\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_{25} \\
&\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \\
&\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}.
\end{aligned}$$

6. [Saracino, #11.17 and 11.18] Let G be a group, and let $H \leq G$.

(a) If G is abelian, prove that G/H is abelian. [**Hint:** You may want to use a result from the last homework assignment.]

Proof. You proved on the last assignment that any homomorphic image of an abelian group is abelian. Therefore, consider the canonical map $\pi : G \to G/H$ given by $\pi(a) = Ha$ (as defined in Problem 1). This is onto, and since G is abelian, G/H must also be abelian.

This can also be proven more directly. Suppose that $Ha, Hb \in G/H$. Then

$$(Ha)(Hb) = Hab = Hba = (Hb)(Ha),$$

so G/H is abelian.

(b) Prove that if G is cyclic, then G/H is also cyclic.

Proof. Suppose that G is cyclic, and let $a \in G$ be a generator for G. We claim that the coset Ha generates G/H. It suffices to show that if $Hb \in G/H$ is any right coset of H, then we have $Hb = (Ha)^m$ for some $m \in \mathbb{Z}$. Well, $b \in G = \langle a \rangle$, so there exists an integer m such that $b = a^m$. Then we have

$$(Ha)^m = \underbrace{(Ha)(Ha)\cdots(Ha)}_{m \text{ times}} = Ha^m = Hb.$$

Since Hb was arbitrary, we see that $G/H = \langle Ha \rangle$, so the quotient group is cyclic.

7. Prove that if G_1 and G_2 are abelian groups, then $G_1 \times G_2$ is abelian.

Proof. Let $(a_1, a_2), (b_1, b_2) \in G_1 \times G_2$. Then

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2) = (b_1 a_1, b_2 a_2) = (b_1, b_2)(a_1, a_2),$$

since both G_1 and G_2 are abelian. Therefore, $G_1 \times G_2$ is abelian.

8. If G_1 and G_2 are groups, prove that $G_1 \times G_2 \cong G_2 \times G_1$.

Proof. Define $\varphi : G_1 \times G_2 \to G_2 \times G_1$ by $\varphi(a_1, a_2) = (a_2, a_1)$. Then φ is a homomorphism, since if we take $(a_1, a_2), (b_1, b_2) \in G_1 \times G_2$, then

$$\begin{aligned}
\varphi((a_1, a_2)(b_1, b_2)) &= \varphi(a_1 b_1, a_2 b_2) \\
&= (a_2 b_2, a_1 b_1) \\
&= (a_2, a_1)(b_2, b_1) \\
&= \varphi(a_1, a_2)\varphi(b_1, b_2).
\end{aligned}$$

Now observe that φ is onto, since given $(a_2, a_1) \in G_2 \times G_1$, $\varphi(a_1, a_2) = (a_2, a_1)$. Finally, it is easy to see that $\varphi(a_1, a_2) = (e_2, e_1)$ if and only if $a_1 = e_1$ and $a_2 = e_2$. Therefore, $\ker \varphi = \{e\}$, and φ is an isomorphism.

9. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_6$, and let $H = \langle (1, 0) \rangle$. Find all the right cosets of H in G, and compute the quotient group G/H. (That is, identify it with a more familiar group.)

Proof. First note that the subgroup H consists of the elements

$$H = \{(0, 0), (1, 0), (2, 0), (3, 0)\}.$$

Therefore, the right cosets of H are

$$\begin{aligned}
H + (0, 0) &= \{(0, 0), (1, 0), (2, 0), (3, 0)\} \\
H + (0, 1) &= \{(0, 1), (1, 1), (2, 1), (3, 1)\} \\
H + (0, 2) &= \{(0, 2), (1, 2), (2, 2), (3, 2)\} \\
H + (0, 3) &= \{(0, 3), (1, 3), (2, 3), (3, 3)\} \\
H + (0, 4) &= \{(0, 4), (1, 4), (2, 4), (3, 4)\} \\
H + (0, 5) &= \{(0, 5), (1, 5), (2, 5), (3, 5)\}
\end{aligned}$$

and you can check that every element of G appears on this list. Therefore, there are six cosets, and it is not hard to see that the group operation on G/H looks exactly like addition mod 6. Therefore, $G/H \cong \mathbb{Z}_6$. (To be more rigorous, you could observe that G/H is an abelian group of order 6, and the Fundamental Theorem of Finite Abelian Groups tells us that there is only one up to isomorphism, namely $\mathbb{Z}_6$.)